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LETTER TO THE EDITOR

On the complete integrability of the Hirota–Satsuma system

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Abstract. We have applied the Painlevé Test to the coupled nonlinear system advocated by Hirota and Satsuma. We have also made a Lie point symmetry analysis of these equations and have shown that the reduced ordinary nonlinear equations are not members of the Painlevé class. Also the Painlevé test itself is seen to fail, so that on both counts the equations are not completely integrable in the usual sense.

In a recent paper Oevel (1982) has investigated the hereditary symmetry structure and the infinite connecting symmetries associated with the nonlinear partial differential system proposed by Hirota and Satsuma (1981). There he raised the point that the Hirota–Satsuma set of equations may not be completely integrable, although they do possess many features exhibited by other such equations. Inspired by this, we ventured to apply the Painlevé test devised by Weiss *et al* (1983) to check whether the system actually conforms to the criterion of the test. We have also made a Lie point symmetry analysis (Bluman and Cole 1974) to see whether the similarity variables reduce the nonlinear partial differential system to the Painlevé class of ordinary nonlinear equations. Finally, we observe that both procedures indicate the lack of complete integrability of the system.

The equations under consideration read as follows

$$\begin{aligned} u_t &= a(u_{xxx}+6uu_x)+2bvv_x \\ v_t &= -v_{xxx}-3uv_x. \end{aligned} \tag{1}$$

To proceed with the Painlevé test, we set

$$u=\sum u_j\phi^{j+\alpha}; \qquad v=\sum v_j\phi^{j+\beta} \tag{2}$$

where we are searching for the singular solution manifold given by $\phi(x, t)=0$. Substitution of (2) in (1) yields

$$\alpha=\beta=-2$$

and

$$\phi^{m-2}\Big[\sum v_{m,t}+\sum v_{m,xxx}\Big]+\phi^{m-3}\Big[\sum(m-2)v_m\phi_t+3\sum(m-2)v_{m,xx}$$
$$+3\sum(m-2)(m-3)v_{mx}\phi_x^2+3\sum(m-2)v_{mx}\phi_{xx}+\sum(m-2)v_m\phi_{xxx}\Big]$$

$$+\sum (m-2)(m-3)(m-4) v_m \phi_x^3 \phi^{m-5}$$
$$+3\sum (m-2)(m-3) v_m \phi_x \phi_{xx} \phi^{m-4} + 3\sum u_j v_{mx} \phi^{j+m-4}$$
$$+3\sum (m-2) u_j v_m \phi^{j+m-5} \phi_x = 0 \tag{3}$$

along with

$$\phi^{j-2}\Big[\sum u_{jt} - a\sum u_{j,xxx}\Big] + \phi^{j-3}\Big[\sum (j-2) u_j \phi_t - 3a\sum (j-2) u_{j,xx}$$
$$-3a\sum (j-2)(j-3) u_{j,x}\phi_x^2 - 3a\sum (j-2) u_{j,x}\phi_{xx} - a\sum (j-2) u_j \phi_{xxx}\Big]$$
$$-a\sum (j-2)(j-3)(j-4) u_j \phi_x^3 \phi^{j-5} - 3a\sum (j-2)(j-3) u_j \phi_x \phi_{xx}\phi^{j-4}$$
$$-6a\sum u_j u_{m,x}\phi^{j+m-4} - 6a\sum u_j (m-2) u_m \phi^{j+m-5}\phi_x$$
$$-2b\sum v_j v_{m,x}\phi^{j+m-4} - 2b\sum v_j (m-2) v_m \phi^{j+m-5}\phi_x = 0. \tag{4}$$

Equations (3) and (4), on equating different powers of ϕ, yield

$$u_0 = -4\phi_x^2 \qquad v_0 = \alpha\phi_x^2$$
$$u_1 = -\tfrac{8}{7}\phi_{xx} \qquad v_1 = -\tfrac{1}{7}\alpha\phi_{xx}. \tag{5}$$

From the lower-order terms we get

$$3\phi_x^3 u_2 + 6\phi_{xx}^2 + 6\phi_x\phi_{xxx} + \phi_x^2\phi_t - 18\phi_x^3\phi_{xx} + \tfrac{163}{13}\phi_x^2\phi_{xxx} - \tfrac{993}{169}\phi_{xx}^2\phi_x = 0, \tag{6}$$

where u_2 is still undetermined. If we collect further terms, equation (3) yields

$$6\alpha\phi_x^3 u_3 + 12\phi_x^3 v_3 + 12\phi_x^2 v_{2,x} - 2\alpha\phi_x\phi_{xt} - \tfrac{514}{169}\alpha\phi_{xx}\phi_{xxx} - 2\alpha\phi_x\phi_{xxxx}$$
$$+\tfrac{25}{13}\alpha\phi_{xx}\phi_t + \tfrac{75}{13}\alpha\phi_{xxxx} - \tfrac{150}{13}\alpha\phi_x^2 - \tfrac{3}{13}\alpha\phi_x\phi_{xx}u_2 = 0 \tag{7}$$

Similar considerations of equation (4) gives rise to

$$12a\phi_x^3 - b\alpha\phi_x^3 v_2 - 2\phi_x^2\phi_t + 12\phi_x^2 + 12\phi_x\phi_{xxx} - 36a\phi_x^3\phi_{xxx} - 36a\phi_x^3\phi_{xx}$$
$$+\tfrac{237}{169}a\phi_{xx}^2\phi_x - \tfrac{127}{13}a\phi_x^2\phi_{xx} = 0 \tag{8}$$

and

$$12a\phi_x^3 u_3 - b\alpha\phi_x^3 v_3 - \tfrac{348}{13}a\phi_x\phi_{xx}u_2 + \tfrac{1}{13}b\alpha\phi_x\phi_{xx}v_2 - 12a\phi_x^2 u_{2,x} + b\alpha\phi_x^2 v_{2,x} + 4\phi_x\phi_{x,t}$$
$$+\tfrac{6}{13}\phi_{xx}\phi_t - \phi_{xx}\phi_{xxx}\lambda - 4\phi_x\phi_{xxxx} - \tfrac{18}{13}a\phi_{xxxx} + \tfrac{36}{13}a\phi_{xxx}\phi_x^2 = 0. \tag{9}$$

If we now apply the usual technique of determination of the 'resonances' by cutting off series (2) we set $u_3 = u_4 = v_3 = v_4 = 0$ then u_2, v_2 will have to satisfy following sets of simultaneous partial differential equations;

$$u_{2t} = a(u_{2xxx} + 6u_2u_{2x}) + 2bv_2v_{2x}$$
$$v_{2t} = -v_{2xxx} - 3u_2v_{2x}$$

and

$$3\phi_x^3 u_2 + 6\phi_{xx}^2 + 6\phi_x\phi_{xxx} + \phi_x^2\phi_t - 18\phi_x^3\phi_{xx} + \tfrac{163}{13}\phi_x^2\phi_{xxx} - \tfrac{993}{169}\phi_{xx}^2\phi_x = 0$$

$$12a\phi_x^3 u_2 - b\alpha\phi_x^3 v_2 - 2\phi_x^2\phi_t + 12\phi_{xx}^2 + 12\phi_x\phi_{xxx} - 36a\phi_x^3\phi_{xx}$$
$$+\tfrac{237}{169}a\phi_{xx}^2\phi_x - \tfrac{127}{13}a\phi_x^2\phi_{xxx} = 0$$
$$-\tfrac{348}{13}a\phi_x\phi_{xx}u_2 + \tfrac{1}{13}b\alpha\phi_x\phi_{xx}v_2 - 12a\phi_x^2 u_{2,x} + b\alpha\phi_x^2 v_{2,x} + 4\phi_x\phi_{x,t}$$
$$+\tfrac{6}{13}\phi_{xx}\phi_t - \phi_{xx}\phi_{xxx}\lambda - 4\phi_x\phi_{xxxx} - \tfrac{18}{13}a\phi_{xxxx} + \tfrac{36}{13}a\phi_{xxx}\phi_x^2 = 0$$
$$12\phi_x^2 v_{2,x} - 2\alpha\phi_x\phi_{x,t} - \tfrac{514}{169}\alpha\phi_{xx}\phi_{xxx} - 2\alpha\phi_x\phi_{xxxx} + \tfrac{25}{13}\alpha\phi_{xx}\phi_t$$
$$+\tfrac{75}{13}\alpha\phi_{xxxx} - \tfrac{150}{13}\alpha\phi_x^2 - \tfrac{3}{13}\alpha\phi_x\phi_{xx}u_2 = 0. \qquad (10)$$

It is now only extremely tedious to check that these equations are not mutually consistent, so that it is impossible to truncate the series for u and v. Even if one does not cut off the series; it is not easy to determine the coefficients u_j, v_j and also it is not possible to get a relation of the form

$$u = \alpha(\partial^2/\partial x^2)(\log\phi) + u_2$$
$$v = \beta(\partial^2/\partial x^2)(\log\varphi) + v_2 \qquad (11)$$

to generate a Bäcklund transformation. So the only conclusion that can be reached is that the Hirota–Satsuma system is not completely integrable even though it possesses infinitely many connecting conservation laws.

To put our assertion on a firm ground, we report here a Lie point symmetry analysis of the set (1) and the subsequent deduction of the similarity variable, in order to check the possibility of reducing system (1) to the original Painlevé class of ordinary nonlinear equations.

Let us consider the following Lie transformation

$$x^* \to x + e\xi(x,t,u,v) \qquad t^* \to t + e\tau(x,t,u,v)$$
$$u^* \to u + e\eta'(x,t,u,v) \qquad v^* \to v + e\eta^2(x,t,u,v) \qquad (12)$$

and demand the invariance of the set (1) which yields:

$$\eta^1 = -2u\alpha \qquad \eta^2 = -2v\alpha$$
$$\xi = \alpha x + b \qquad \tau = ct + d \qquad \text{and} \quad a = +\tfrac{1}{2} \qquad (13)$$

So that symmetry condition automatically fixes the value of the constant a in equation (13) to be $\frac{1}{2}$, as originally considered by Oevel (1982) and Hirota and Satsuma (1981). We now write down the Lagrange equation

$$\mathrm{d}x/(\alpha x + b) = \mathrm{d}t/(ct+d) = \mathrm{d}u/-2u\alpha = \mathrm{d}v/-2v\alpha, \qquad (14)$$

which upon integration yields the following similarity forms

$$u = \frac{1}{(\alpha x + b)^2} f\left(\frac{x}{3t^{1/3}}\right) \qquad v = \frac{1}{(\alpha x + b)^2} g\left(\frac{x}{4t^{1/3}}\right) \qquad (15)$$

Changing to these functions we find that f and g satisfy the following ordinary nonlinear differential equations

$$\mathrm{d}^3 g/\mathrm{d}\sigma^3 - 6(\mathrm{d}^2 g/\mathrm{d}\sigma^2) + 9(2\sigma^2 - 1/\sigma)\,\mathrm{d}g/\mathrm{d}\sigma + 27(\mathrm{d}g/\mathrm{d}\sigma)h\sigma^2 - 27(\tfrac{8}{9} + 2h)g\sigma^3 = 0 \qquad (16)$$

$$a\sigma^3\,\mathrm{d}^3 f/\mathrm{d}\sigma^3 - 18a\sigma^2\,\mathrm{d}^2 f/\mathrm{d}\sigma^2 + 54a\sigma\,\mathrm{d}f/\mathrm{d}\sigma - 216af + (\mathrm{d}f/\mathrm{d}\sigma)\sigma^4$$
$$+27\sigma(6aff' + 2bgg') - 162(6af^2 + 2bg^2) = 0. \qquad (17)$$

It is now a matter of routine calculation following Ince (1956) to see that this set of equations does not belong to the Painlevé class classified there. So we have demonstrated in the above discussions that some nonlinear equations, even though they possess an inverse scattering transform and infinite number of commuting integrals, may not be completely integrable.

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